Massless Dirac equation as a special case of Cosserat elasticity

Olga Chervova* and Dmitri Vassiliev†

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Abstract

We suggest an alternative mathematical model for the massless neutrino. Consider an elastic continuum in 3-dimensional Euclidean space and assume that points of this continuum can experience no displacements, only rotations. This framework is a special case of the so-called Cosserat theory of elasticity. Rotations of points of the continuum are described by attaching to each point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables (unknowns) of our theory we choose a coframe and a density. We write down a potential energy which is conformally invariant and then incorporate time in the standard Newtonian way, by subtracting kinetic energy. Finally, we rewrite the resulting nonlinear variational problem in terms of an unknown spinor field. We look for quasi-stationary solutions, i.e. solutions that harmonically oscillate in time. We prove that in the quasi-stationary setting our model is equivalent to a pair of massless Dirac equations. The crucial element of the proof is the observation that our Lagrangian admits a factorisation.

1 Introduction

The massless Dirac equation is a system of two homogeneous linear complex partial differential equations for two complex unknowns. The unknowns (components of a spinor) are functions of time and the three spatial coordinates. This equation is the accepted mathematical model for the massless neutrino.

The geometric interpretation of the massless Dirac equation is rather complicated. It relies on the use of notions such as

- spinor,
- Pauli matrices,
- covariant derivative (note that formula (5) for the covariant derivative of a spinor field is quite tricky).

*Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK; olgac@math.ucl.ac.uk.

†Department of Mathematics and Institute of Origins, University College London, Gower Street, London WC1E 6BT, UK; D.Vassiliev@ucl.ac.uk.

There is also a logical problem with the massless Dirac equation in that it predicts the existence of four essentially different types of plane wave solutions which are called left-handed neutrino, right-handed neutrino, left-handed antineutrino and right-handed antineutrino. Only left-handed neutrinos and right-handed antineutrinos are observed experimentally.

The purpose of this paper is to formulate an alternative mathematical model for the massless neutrino, a model which is geometrically much simpler. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a

- metric,
- differential form,
- wedge product,
- exterior derivative.

Our model also overcomes the logical problem mentioned in the previous paragraph in that it predicts the existence of only two essentially different types of plane wave solutions. These correspond to clockwise and anticlockwise rotations of the coframe.

The paper has the following structure. In Section 2 we introduce our notation and in Section 3 we formulate the massless Dirac equation. In Section 4 we formulate our mathematical model. In Section 5 we rewrite our Lagrangian in relativistic form; we do this to show that our Lagrangian looks simpler in relativistic notation, though we do not pursue the relativistic approach consistently in the current paper. In Section 6 we translate our model into the language of spinors. In Section 7 we prove Theorem 1 which is the main result of the paper: this theorem establishes that in the quasi-stationary case (prescribed oscillation in time with frequency ω) our mathematical model is equivalent to a pair of massless Dirac equations. The crucial element of the proof of Theorem 1 is the observation that our Lagrangian admits a factorisation; this factorisation is the subject of Lemma 2. Section 8 deals with plane wave solutions. The concluding discussion is contained in Section 9.

2 Notation and conventions

We work on a 3-manifold M equipped with prescribed **negative** (i.e. with signature $---$) metric $g_{\alpha\beta}$. We choose negative metric on the 3-manifold M in order to facilitate the subsequent extension (see Section 5) to a Lorentzian metric

$$\mathbf{g}_{\boldsymbol{\alpha\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix} \tag{1}$$

of signature $+---$ on the 4-manifold $\mathbb{R}\times M$. We denote time by x^0 and local coordinates on M by x^α, $\alpha = 1, 2, 3$.

All constructions presented in the paper are local so we do not make a priori assumptions on the geometric structure of $\{M,g\}$.

Our notation follows [1, 2, 3]. In particular, in line with the traditions of particle physics, we use Greek letters to denote tensor (holonomic) indices. The only difference with references [1, 2, 3] is that, by default, we assume tensor indices to run through the values 1, 2, 3 rather than 0, 1, 2, 3. The index 0 will be treated separately.

Details of our spinor notation are given in Appendix A of [1]. We choose the zeroth Pauli matrix to be the identity matrix,

$$\sigma^0{}_{a\dot b}=\sigma^{0a\dot b}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{2}$$

The defining relations for the three remaining Pauli matrices are

$$\sigma^\alpha{}_{a\dot b}\sigma^{\beta c\dot b}+\sigma^\beta{}_{a\dot b}\sigma^{\alpha c\dot b}=2g^{\alpha\beta}\delta_a{}^c,\qquad \alpha,\beta=1,2,3, \tag{3}$$

$$\sigma^\alpha{}_{a\dot b}\sigma^{0a\dot b}=0,\qquad \alpha=1,2,3. \tag{4}$$

We assume that all our Pauli matrices do not depend on time x^0. Formulae (2)–(4) mean that we are effectively working on the 4-manifold $\mathbb{R}\times M$ equipped with Lorentzian metric (1).

By ∇ we denote the covariant derivative on the 3-manifold M with respect to the Levi-Civita connection. It acts on a vector field and a spinor field as $\nabla_\alpha v^\beta:=\partial_\alpha v^\beta+\Gamma^\beta{}_{\alpha\gamma}v^\gamma$ and

$$\nabla_\alpha\xi^a:=\partial_\alpha\xi^a+\frac{1}{4}\sigma_\beta{}^{a\dot c}(\partial_\alpha\sigma^\beta{}_{b\dot c}+\Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{b\dot c})\xi^b \tag{5}$$

respectively, where $\partial_\alpha:=\partial/\partial x^\alpha$ and

$$\Gamma^\beta{}_{\alpha\gamma}=\left\{\begin{matrix}\beta\\ \alpha\gamma\end{matrix}\right\}:=\frac{1}{2}g^{\beta\delta}(\partial_\alpha g_{\gamma\delta}+\partial_\gamma g_{\alpha\delta}-\partial_\delta g_{\alpha\gamma})$$

are the Christoffel symbols. We also denote $\partial_0:=\partial/\partial x^0$.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors P and Q of rank r we define their dot product as $P\cdot Q:=\frac{1}{r!}P_{\alpha_1\ldots\alpha_r}Q_{\beta_1\ldots\beta_r}g^{\alpha_1\beta_1}\ldots g^{\alpha_r\beta_r}$. We also define $\|P\|^2:=P\cdot P$.

3 The Dirac equation

The following system of two complex linear partial differential equations on $\mathbb{R}\times M$ for two complex unknowns is known as the *massless Dirac equation*:

$$i(\pm\sigma^0{}_{a\dot b}\partial_0+\sigma^\alpha{}_{a\dot b}\nabla_\alpha)\xi^a=0. \tag{6}$$

Here ξ is a spinor field which plays the role of dynamical variable (unknown quantity) and is a function of time $x^0\in\mathbb{R}$ and local coordinates (x^1,x^2,x^3) on

the 3-manifold M. Summation in (6) is carried out over $\alpha = 1, 2, 3$. The two choices of sign in (6) give two versions of the massless Dirac equation which differ by time reversal. Thus, we have a pair of massless Dirac equations.

The corresponding Lagrangian density is

$$L^{\pm}_{\mathrm{Dir}}(\xi) := \frac{i}{2}\big[\pm(\bar{\xi}^{\dot b}\sigma^0{}_{a\dot b}\partial_0\xi^a - \xi^a\sigma^0{}_{a\dot b}\partial_0\bar{\xi}^{\dot b}) + (\bar{\xi}^{\dot b}\sigma^\alpha{}_{a\dot b}\nabla_\alpha\xi^a - \xi^a\sigma^\alpha{}_{a\dot b}\nabla_\alpha\bar{\xi}^{\dot b})\big]\sqrt{|\det g|}\,. \quad (7)$$

Note that the massless Dirac equation and Lagrangian are often called *Weyl* equation and Lagrangian respectively.

4 Our model

A *coframe* ϑ is a triplet of real covector fields $\vartheta^j \in T^*M$, $j = 1, 2, 3$, satisfying the constraint

$$g = -\vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3. \quad (8)$$

For the sake of clarity we repeat formula (8) giving the tensor indices explicitly: $g_{\alpha\beta} = -\vartheta^1_\alpha\vartheta^1_\beta - \vartheta^2_\alpha\vartheta^2_\beta - \vartheta^3_\alpha\vartheta^3_\beta$. We assume our coframe to be right-handed, i.e. we assume that $\det\vartheta^j_\alpha > 0$.

Formula (8) means that the coframe is a field of orthonormal bases. Of course, at every point of the 3-manifold M the choice of coframe is not unique: there are 3 real degrees of freedom in choosing the coframe and any pair of coframes is related by an orthogonal transformation.

As dynamical variables in our model we choose a coframe ϑ and a positive density ρ. These live on the 3-manifold M and are functions of local coordinates (x^1, x^2, x^3) as well as of time $x^0 \in \mathbb{R}$.

At a physical level choosing the coframe as an unknown quantity means that we view our 3-manifold M as an elastic continuum and allow every point of this continuum to rotate, assuming that rotations of different points are totally independent. These rotations are described mathematically by attaching to each point a coframe (= orthonormal basis). The approach in which the coframe plays the role of a dynamical variable is known as *teleparallelism* (= absolute parallelism = fernparallelismus). This is a subject promoted by A. Einstein and É. Cartan [4, 5, 6].

The idea of rotating points may seem exotic, however it has long been accepted in continuum mechanics within the so-called Cosserat theory of elasticity [7]. The Cosserat theory of elasticity has been in existence since 1909 and appears under various names in modern applied mathematics literature such as *oriented medium*, *asymmetric elasticity*, *micropolar elasticity*, *micromorphic elasticity*, *moment elasticity* etc. Cosserat elasticity is closely related to the theory of ferromagnetic materials [8] and the theory of liquid crystals [9, 10]. As to teleparallelism, it is, effectively, a special case of Cosserat elasticity: here the assumption is that the elastic continuum experiences no displacements, only rotations. It is interesting to note that when Cartan started developing what

eventually became modern differential geometry he acknowledged [11] that he drew inspiration from the 'beautiful' work of the Cosserat brothers.

We define the 3-form

$$T^{\mathrm{ax}} := -\frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3) \tag{9}$$

where d denotes the exterior derivative. This 3-form is called *axial torsion of the teleparallel connection*. An explanation of the geometric meaning of the latter phrase as well as a detailed exposition of the application of torsion in field theory and the history of the subject can be found in [12]. For our purposes the 3-form (9) is simply a measure of deformations generated by rotations of points of the 3-manifold M.

Note that the 3-form (9) has the remarkable property of conformal covariance: if we rescale our coframe as

$$\vartheta^j \mapsto e^h \vartheta^j \tag{10}$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function, then our metric is scaled as

$$g_{\alpha\beta} \mapsto e^{2h} g_{\alpha\beta} \tag{11}$$

and our 3-form is scaled as

$$T^{\mathrm{ax}} \mapsto e^{2h} T^{\mathrm{ax}} \tag{12}$$

without the derivatives of h appearing.

We take the potential energy of our continuum to be

$$P(x^0) := -\int_M \|T^{\mathrm{ax}}\|^2 \rho\, dx^1 dx^2 dx^3. \tag{13}$$

We put a minus sign in the RHS of (13) because the metric on our 3-manifold M is negative, see beginning of Section 2. As a result, our potential energy is nonnegative, as it should be.

It is easy to see that the potential energy (13) is conformally invariant: it does not change if we rescale our coframe as (10) and our density as

$$\rho \mapsto e^{2h} \rho. \tag{14}$$

This follows from formulae (12), (11) and $\|T^{\mathrm{ax}}\|^2 = \frac{1}{3!} T^{\mathrm{ax}}_{\alpha\beta\gamma} T^{\mathrm{ax}}_{\kappa\lambda\mu} g^{\alpha\kappa} g^{\beta\lambda} g^{\gamma\mu}$. Thus, the guiding principle in our choice of potential energy (13) is conformal invariance.

We take the kinetic energy of our continuum to be

$$K(x^0) := \int_M \|\dot\vartheta\|^2 \rho\, dx^1 dx^2 dx^3 \tag{15}$$

where $\dot\vartheta$ is the 2-form

$$\dot\vartheta := \frac{1}{3}(\vartheta^1 \wedge \partial_0 \vartheta^1 + \vartheta^2 \wedge \partial_0 \vartheta^2 + \vartheta^3 \wedge \partial_0 \vartheta^3). \tag{16}$$

Unlike (13), we did not put a minus sign in the RHS of (15): this is because we are now squaring a 2-form rather than a 3-form (the number 2 is even whereas the number 3 is odd). As a result, our kinetic energy is nonnegative, as it should be.

It is easy to see that the 2-form (16) is, up to a constant factor, the Hodge dual of angular velocity, so (15) is the standard expression for the kinetic energy of a homogeneous isotropic Cosserat continuum. One should think of a collection of identical infinitesimal rotating solid balls distributed with density ρ. We think in terms of balls rather than ellipsoids because of the isotropy, i.e. we do not have preferred axes of rotation.

We now combine the potential energy (13) and kinetic energy (15) to form the action (variational functional) of our dynamic problem:

$$S := \int_{\mathbb{R}} (K(x^0) - P(x^0))\, dx^0 = \int_{\mathbb{R}\times M} L(\vartheta,\rho)\, dx^0 dx^1 dx^2 dx^3 \qquad (17)$$

where

$$L(\vartheta,\rho) := (\|\dot\vartheta\|^2 + \|T^{\mathrm{ax}}\|^2)\rho \qquad (18)$$

is our Lagrangian density. Note that this Lagrangian density is conformally invariant in the Lorentzian sense. The latter means that we rescale time simultaneously with the rescaling of the 3-dimensional coframe.

Our field equations (Euler–Lagrange equations) are obtained by varying the action (17) with respect to the coframe ϑ and density ρ. Varying with respect to the density ρ is easy: this gives the field equation $\|\dot\vartheta\|^2 + \|T^{\mathrm{ax}}\|^2 = 0$ which is equivalent to $L(\vartheta,\rho) = 0$. Varying with respect to the coframe ϑ is more difficult because we have to maintain the metric constraint (8); recall that the metric is assumed to be prescribed (fixed).

We do not write down the field equations for the Lagrangian density $L(\vartheta,\rho)$ explicitly. We note only that they are highly nonlinear and do not appear to bear any resemblance to the linear Dirac equation (6).

5 Relativistic representation of our Lagrangian

In this section we work on the 4-dimensional manifold $\mathbb{R}\times M$ equipped with Lorentzian metric (1). This manifold is an extension of the original 3-manifold M. We use **bold** type for extended quantities.

We extend our coframe as

$$\boldsymbol{\vartheta}^0_{\boldsymbol{\alpha}} = \begin{pmatrix} 1 \\ 0_\alpha \end{pmatrix}, \qquad (19)$$

$$\boldsymbol{\vartheta}^j_{\boldsymbol{\alpha}} = \begin{pmatrix} 0 \\ \vartheta^j_\alpha \end{pmatrix}, \qquad j = 1,2,3, \qquad (20)$$

where the bold tensor index $\boldsymbol{\alpha}$ runs through the values 0, 1, 2, 3, whereas its non-bold counterpart α runs through the values 1, 2, 3. In particular, the 0_α in formula (19) stands for a column of three zeros.

The extended metric (1) is expressed via the extended coframe (19), (20) as

$$\mathbf{g} = \boldsymbol{\vartheta}^0 \otimes \boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \otimes \boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \otimes \boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \otimes \boldsymbol{\vartheta}^3 \tag{21}$$

(compare with (8)). The extended axial torsion is

$$\mathbf{T}^{\mathrm{ax}} := \frac{1}{3}(\underbrace{\boldsymbol{\vartheta}^0 \wedge d\boldsymbol{\vartheta}^0}_{=0} - \boldsymbol{\vartheta}^1 \wedge d\boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \wedge d\boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \wedge d\boldsymbol{\vartheta}^3) \tag{22}$$

where d denotes the exterior derivative on $\mathbb{R} \times M$. Formula (22) can be rewritten as

$$\mathbf{T}^{\mathrm{ax}} = \boldsymbol{\vartheta}^0 \wedge \dot{\vartheta} + T^{\mathrm{ax}} \tag{23}$$

with $\dot{\vartheta}$ and T^{ax} defined by (16) and (9) respectively. Squaring (23) we get $\|\mathbf{T}^{\mathrm{ax}}\|^2 = \|\dot{\vartheta}\|^2 + \|T^{\mathrm{ax}}\|^2$ which implies that our Lagrangian density (18) can be rewritten as

$$L(\vartheta, \rho) = \|\mathbf{T}^{\mathrm{ax}}\|^2 \rho. \tag{24}$$

The point of the arguments presented in this section was to show that if one accepts the relativistic point of view then our Lagrangian density (18) takes the especially simple form (24). A consistent pursuit of the relativistic approach would require the variation of all four elements of the extended coframe which we do not do in the current paper. Instead, we assume that the zeroth element of the extended coframe is specified by formula (19).

6 Choosing a common language

In order to compare the two models described in Sections 3 and 4 we need to choose a common mathematical language. We choose the language of spinors. Namely, we express the coframe and density via a spinor field ξ^a according to formulae

$$s = \xi^a \sigma^0{}_{a\dot{b}} \bar{\xi}^{\dot{b}}, \tag{25}$$

$$\rho = s\,\sqrt{|\det g|}\,, \tag{26}$$

$$(\vartheta^1 + i\vartheta^2)_\alpha = -s^{-1} \xi^a \sigma_{\alpha a\dot{b}} \epsilon^{\dot{b}\dot{c}} \sigma^0{}_{d\dot{c}} \xi^d, \tag{27}$$

$$\vartheta^3_\alpha = s^{-1} \xi^a \sigma_{\alpha a\dot{b}} \bar{\xi}^{\dot{b}} \tag{28}$$

where

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

Note that throughout this paper we assume that the density ρ does not vanish. This is equivalent to the assumption that the spinor field ξ^a does not vanish.

Formulae (25)–(29) are effectively a special case of those from Section 5 of [13], the only difference being that now the two spinors in the bispinor are not independent but related as $\eta_{\dot{b}} = \sigma^0{}_{a\dot{b}} \xi^a$. This is hardly surprising as in the current paper we work in a non-relativistic 3-dimensional setting so we do not

really need bispinors. We also do not really need to distinguish between dotted and undotted spinor indices but we retained this distinction in order to facilitate comparison with [13].

Formulae (25)–(29) establish a one-to-two correspondence between a coframe ϑ and a (positive) density ρ on the one hand and a nonvanishing spinor field ξ^a on the other. The correspondence is one-to-two because for given ϑ and ρ the above formulae define ξ^a uniquely up to choice of sign. This is in agreement with the generally accepted view (see, for example, Section 19 in [14] or Section 3.5 in [15]) that the sign of a spinor does not have a physical meaning.

Remark 1 *Formulae (25)–(29) look somewhat unnatural in that they are assign a special meaning to the element ϑ^3 of our coframe. This can be overcome by allowing rigid rotations of the coframe, i.e. linear transformations $\vartheta^j \mapsto O^j{}_k\vartheta^k$ where $O^j{}_k$ $j,k=1,2,3$, is a constant orthogonal matrix with determinant $+1$. Note that such transformations are totally unrelated to coordinate transformations. It is well know that axial torsion is invariant under rigid rotations of the coframe, hence our model described in Section 4 is invariant under rigid rotations of the coframe. In particular, it is natural to view coframes which differ by a rigid rotation as equivalent.*

7 Quasi-stationary case

For both models, the traditional one (described in Section 3) and our model (described in Section 4), we shall now seek solutions of the form

$$\xi^a(x^0,x^1,x^2,x^3)=e^{-i\omega x^0}\zeta^a(x^1,x^2,x^3) \tag{30}$$

where $\omega \neq 0$ is a fixed real number. We shall call solutions of the type (30) *quasi-stationary*. In seeking quasi-stationary solutions what we are doing is separating out the time variable x^0, as is done when reducing, say, the wave equation to the Helmholtz equation.

Substituting (30) into (7) we get

$$L^\pm_{\mathrm{Dir}}(\zeta)=\left[\frac{i}{2}(\bar\zeta^{\dot b}\sigma^\alpha{}_{a\dot b}\nabla_\alpha\zeta^a-\zeta^a\sigma^\alpha{}_{a\dot b}\nabla_\alpha\bar\zeta^{\dot b})\pm\omega\zeta^a\sigma^0{}_{a\dot b}\bar\zeta^{\dot b}\right]\sqrt{|\det g|}\,. \tag{31}$$

Substituting (30) into (25)–(29) and the latter into (16) we get $\dot\vartheta=\frac{4}{3}\omega\vartheta^1\wedge\vartheta^2$. Hence, $\|\dot\vartheta\|^2=\frac{16}{9}\omega^2$ and formula (18) becomes

$$L(\zeta)=\left(\|T^{\mathrm{ax}}\|^2+\frac{16}{9}\omega^2\right)\rho \tag{32}$$

where

$$\rho=\zeta^a\sigma^0{}_{a\dot b}\bar\zeta^{\dot b}\sqrt{|\det g|}\,. \tag{33}$$

Note that our 3-dimensional metric is negative (see beginning of Section 2), so $\|T^{\mathrm{ax}}\|^2\le 0$ and the Lagrangian density (32) may vanish. In fact, as we shall

see from the proof of Theorem 1 in the end of this section, it has to vanish on solutions of our field equations.

In order to compare the Lagrangian densities (31) and (32) we need an explicit formula for T^{ax} in terms of the spinor field. This formula is given in the following

Lemma 1 *We have*

$$(*T^{\mathrm{ax}})\rho = \frac{2i}{3}(\bar{\xi}^{\dot{b}}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a} - \xi^{a}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\bar{\xi}^{\dot{b}})\sqrt{|\det g|} \tag{34}$$

where

$$*\,T^{\mathrm{ax}} := \frac{1}{3!}\,\sqrt{|\det g|}\,(T^{\mathrm{ax}})^{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} \tag{35}$$

is the Hodge dual of T^{ax}*,* $\varepsilon_{123} := +1$*.*

Proof Observe that time does not appear in the formula (9) for axial torsion (summation is carried out over $\alpha = 1, 2, 3$). Hence the result we are proving should hold for spinor fields ξ with arbitrary dependence on time and not only for quasi-stationary ones. As the expression in the RHS of (34) is an invariant we can also temporarily (for the duration of the proof of Lemma 1) suspend our convention (Section 2) that our Pauli matrices do not depend on time.

In order to simplify calculations we observe that we have freedom in our choice of Pauli matrices. It is sufficient to prove formula (34) for one particular choice of Pauli matrices, hence it is natural to choose Pauli matrices in a way that makes calculations as simple as possible. Note that this trick is not new: it was, for example, extensively used by A. Dimakis and F. Müller-Hoissen [16, 17, 18].

We choose Pauli matrices

$$\sigma_{\alpha a\dot{b}} = \vartheta^{j}_{\alpha}\, s_{ja\dot{b}} = \vartheta^{1}_{\alpha}\, s_{1a\dot{b}} + \vartheta^{2}_{\alpha}\, s_{2a\dot{b}} + \vartheta^{3}_{\alpha}\, s_{3a\dot{b}} \tag{36}$$

where

$$s_{ja\dot{b}} = \begin{pmatrix} s_{1a\dot{b}} \\ s_{2a\dot{b}} \\ s_{3a\dot{b}} \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}. \tag{37}$$

Let us stress that in the statement of Lemma 1 Pauli matrices are not assumed to be related in any way to the coframe ϑ. We are just choosing the particular Pauli matrices (36), (37) to simplify calculations in our proof.

Substituting (36), (37) into (27), (28) we see that the system (25)–(29) can be easily resolved for ξ: solutions are spinors with $\xi^2 = 0$ and ξ^1 which is nonzero and real. Thus, we have

$$\xi^{a} = \pm\begin{pmatrix} e^{h} \\ 0 \end{pmatrix} \tag{38}$$

where $h : M \to \mathbb{R}$ is a scalar function. Formula (38) may seem strange: we are proving Lemma 1 for a general nonvanishing spinor field ξ but ended up with formula (38) which is very specific. However, there is no contradiction here because we chose Pauli matrices specially adapted to the coframe and, hence, specially adapted to the corresponding spinor field.

Formulae (5) and (38) imply

$$\begin{aligned}\frac{i}{2}\bar\xi^{\dot d}\sigma^\alpha{}_{a\dot d}\nabla_\alpha\xi^a &= \frac{i}{2}\bar\xi^{\dot d}(\sigma^\alpha{}_{a\dot d}\partial_\alpha h)\xi^a + \frac{i}{8}\bar\xi^{\dot d}\sigma^\alpha{}_{a\dot d}\sigma_\beta{}^{a\dot c}(\partial_\alpha\sigma^\beta{}_{b\dot c} + \Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{b\dot c})\xi^b\\ &= \frac{ie^{2h}}{8}\sigma^\alpha{}_{a\dot 1}\sigma_\beta{}^{a\dot c}(\partial_\alpha\sigma^\beta{}_{1\dot c} + \Gamma^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{1\dot c}) + \ldots = \frac{is}{8}\sigma^\alpha{}_{a\dot 1}\sigma_\beta{}^{a\dot c}\nabla_\alpha\sigma^\beta{}_{1\dot c} + \ldots\end{aligned}$$

where $s = \xi^a\sigma^0{}_{a\dot b}\bar\xi^{\dot b} = e^{2h}$ is the scalar (25) and the "dots" denote purely imaginary terms. We now write down the spinor summation indices explicitly:

$$\begin{aligned}\frac{i}{2}\bar\xi^{\dot d}\sigma^\alpha{}_{a\dot d}\nabla_\alpha\xi^a = \frac{is}{8}\big[&\sigma^\alpha{}_{1\dot 1}\sigma_\beta{}^{1\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + \sigma^\alpha{}_{1\dot 1}\sigma_\beta{}^{1\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2}\\ &+ \sigma^\alpha{}_{2\dot 1}\sigma_\beta{}^{2\dot 1}\nabla_\alpha\sigma^\beta{}_{1\dot 1} + \sigma^\alpha{}_{2\dot 1}\sigma_\beta{}^{2\dot 2}\nabla_\alpha\sigma^\beta{}_{1\dot 2}\big] + \ldots\end{aligned}$$

Finally, substituting explicit formulae (36), (37) for our Pauli matrices and using the "dots" to absorb all purely imaginary terms we get

$$\begin{aligned}\frac{i}{2}\bar\xi^{\dot d}\sigma^\alpha{}_{a\dot d}\nabla_\alpha\xi^a &= \frac{is}{8}\big[\vartheta^{3\alpha}(-\vartheta^3_\beta)\nabla_\alpha\vartheta^{3\beta} - \vartheta^{3\alpha}(\vartheta^1 - i\vartheta^2)_\beta\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta\\ &\quad - (\vartheta^1 - i\vartheta^2)^\alpha(\vartheta^1 + i\vartheta^2)_\beta\nabla_\alpha\vartheta^{3\beta} + (\vartheta^1 - i\vartheta^2)^\alpha\vartheta^3_\beta\nabla_\alpha(\vartheta^1 + i\vartheta^2)^\beta\big] + \ldots\\ &= \frac{is}{8}\big[-i\vartheta^{3\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{2\beta} + i\vartheta^{3\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{1\beta} - i\vartheta^{1\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{3\beta}\\ &\quad + i\vartheta^{2\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{3\beta} + i\vartheta^{1\alpha}\vartheta^3_\beta\nabla_\alpha\vartheta^{2\beta} - i\vartheta^{2\alpha}\vartheta^3_\beta\nabla_\alpha\vartheta^{1\beta}\big] + \ldots\\ &= \frac{s}{8}\big[\vartheta^{3\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{2\beta} - \vartheta^{3\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{1\beta} + \vartheta^{1\alpha}\vartheta^2_\beta\nabla_\alpha\vartheta^{3\beta}\\ &\quad - \vartheta^{2\alpha}\vartheta^1_\beta\nabla_\alpha\vartheta^{3\beta} - \vartheta^{1\alpha}\vartheta^3_\beta\nabla_\alpha\vartheta^{2\beta} + \vartheta^{2\alpha}\vartheta^3_\beta\nabla_\alpha\vartheta^{1\beta}\big] + \ldots\\ &= \frac{s}{8}[(\vartheta^1\wedge\vartheta^2)\cdot d\vartheta^3 + (\vartheta^3\wedge\vartheta^1)\cdot d\vartheta^2 + (\vartheta^2\wedge\vartheta^3)\cdot d\vartheta^1] + \ldots\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{i}{2}(\bar\xi^{\dot b}\sigma^\alpha{}_{a\dot b}\nabla_\alpha\xi^a - \xi^a\sigma^\alpha{}_{a\dot b}\nabla_\alpha\bar\xi^{\dot b})\\ &\qquad = \frac{s}{4}[(\vartheta^1\wedge\vartheta^2)\cdot d\vartheta^3 + (\vartheta^3\wedge\vartheta^1)\cdot d\vartheta^2 + (\vartheta^2\wedge\vartheta^3)\cdot d\vartheta^1]. \qquad (39)\end{aligned}$$

Axial torsion is defined by formula (9) whereas

$$\sqrt{|\det g|}\,\varepsilon_{\alpha\beta\gamma} = (\vartheta^1\wedge\vartheta^2\wedge\vartheta^3)_{\alpha\beta\gamma}$$

so formula (35) can be rewritten as

$$
\begin{aligned}
*T^{\mathrm{ax}} = T^{\mathrm{ax}} \cdot (\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3) &= -\frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3) \cdot (\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3) \\
&= \frac{1}{3}[(\vartheta^1 \wedge \vartheta^2) \cdot d\vartheta^3 + (\vartheta^3 \wedge \vartheta^1) \cdot d\vartheta^2 + (\vartheta^2 \wedge \vartheta^3) \cdot d\vartheta^1]. \qquad (40)
\end{aligned}
$$

Here we used the fact that because of the negativity of our metric (see (8)) we have $\|\vartheta^j\|^2 = -1$, $j = 1, 2, 3$.

Formulae (40), (39) and (26) imply (34). $\square$

We are now in a position to establish the relationship between the Lagrangian densities (31) and (32).

Lemma 2 *In the quasi-stationary case (30) our Lagrangian density (32) factorises as*

$$
L(\zeta) = -\frac{32\omega}{9} \frac{L^+_{\mathrm{Dir}}(\zeta) L^-_{\mathrm{Dir}}(\zeta)}{L^+_{\mathrm{Dir}}(\zeta) - L^-_{\mathrm{Dir}}(\zeta)} \,. \qquad (41)
$$

Let us emphasise once again that throughout this paper we assume that the density ρ does not vanish. In view of formulae (31), (33), in the quasi-stationary case this assumption can be equivalently rewritten as

$$
L^+_{\mathrm{Dir}}(\zeta) \neq L^-_{\mathrm{Dir}}(\zeta) \qquad (42)
$$

so the denominator in (41) is nonzero.

Proof of Lemma 2 In the quasi-stationary case (30) formula (34) takes the form

$$
(*T^{\mathrm{ax}})\rho = \frac{2i}{3} \big[(\bar{\xi}^{\dot{b}} \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \xi^a - \xi^a \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \bar{\xi}^{\dot{b}})\big] \sqrt{|\det g|}
$$

because the factor $e^{-i\omega x^0}$ cancels out. Hence (recall that our metric is negative)

$$
\|T^{\mathrm{ax}}\|^2 = -\frac{16}{9\rho^2} \left[\frac{i}{2} (\bar{\xi}^{\dot{b}} \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \xi^a - \xi^a \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \bar{\xi}^{\dot{b}}) \sqrt{|\det g|} \right]^2 .
$$

Substituting this into (32) we get

$$
L(\zeta) = -\frac{16}{9\rho} \left(\left[\frac{i}{2} (\bar{\xi}^{\dot{b}} \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \xi^a - \xi^a \sigma^\alpha{}_{a\dot{b}} \nabla_\alpha \bar{\xi}^{\dot{b}}) \sqrt{|\det g|} \right]^2 - \omega^2 \rho^2 \right) . \qquad (43)
$$

Formulae (43), (31) and (33) imply (41). $\square$

The following theorem is the main result of our paper.

Theorem 1 *In the quasi-stationary case (30) a spinor field ζ is a solution of the field equations for the Lagrangian density $L(\zeta)$ if and only if this spinor field is a solution of the field equations for the Lagrangian density $L^+_{\mathrm{Dir}}(\zeta)$ or the field equations for the Lagrangian density $L^-_{\mathrm{Dir}}(\zeta)$.*

Proof Observe that the Dirac Lagrangian densities $L^\pm_{\rm Dir}$ defined by formula (31) possess the property of scaling covariance:

$$L^\pm_{\rm Dir}(e^h\zeta)=e^{2h}L^\pm_{\rm Dir}(\zeta) \tag{44}$$

where $h:M\to\mathbb{R}$ is an arbitrary scalar function. We claim that the statement of the theorem follows from (41) and (44). The proof presented below is an abstract one and does not depend on the physical nature of the dynamical variable ζ, the only requirement being that it is an element of a vector space so that scaling makes sense.

Note that formulae (41) and (44) imply that the Lagrangian density L possesses the property of scaling covariance, so all three of our Lagrangian densities, L, $L^+_{\rm Dir}$ and $L^-_{\rm Dir}$, have this property. Note also that if ζ is a solution of the field equation for some Lagrangian density $\mathcal{L}$ possessing the property of scaling covariance then $\mathcal{L}(\zeta)=0$. Indeed, let us perform a scaling variation of our dynamical variable

$$\zeta\mapsto\zeta+h\zeta \tag{45}$$

where $h:M\to\mathbb{R}$ is an arbitrary "small" scalar function with compact support. Then $0=\delta\int\mathcal{L}(\zeta)=2\int h\mathcal{L}(\zeta)$ which holds for arbitrary h only if $\mathcal{L}(\zeta)=0$.

In the remainder of the proof the variations of ζ are arbitrary and not necessarily of the scaling type (45).

Suppose that ζ is a solution of the field equation for the Lagrangian density $L^+_{\rm Dir}$. [The case when ζ is a solution of the field equation for the Lagrangian density $L^-_{\rm Dir}$ is handled similarly.] Then $L^+_{\rm Dir}(\zeta)=0$ and, in view of (42), $L^-_{\rm Dir}(\zeta)\ne0$. Varying ζ, we get

$$\begin{aligned}\delta\int L(\zeta)=-\frac{32\omega}{9}\Bigl(\int\frac{L^-_{\rm Dir}(\zeta)}{L^+_{\rm Dir}(\zeta)-L^-_{\rm Dir}(\zeta)}\delta L^+_{\rm Dir}(\zeta)+\int L^+_{\rm Dir}(\zeta)\delta\frac{L^-_{\rm Dir}(\zeta)}{L^+_{\rm Dir}(\zeta)-L^-_{\rm Dir}(\zeta)}\Bigr)\\ =\frac{32\omega}{9}\int\delta L^+_{\rm Dir}(\zeta)=\frac{32\omega}{9}\,\delta\int L^+_{\rm Dir}(\zeta)\end{aligned}$$

so

$$\delta\int L(\zeta)=\frac{32\omega}{9}\,\delta\int L^+_{\rm Dir}(\zeta)\,. \tag{46}$$

We assumed that ζ is a solution of the field equation for the Lagrangian density $L^+_{\rm Dir}$ so $\delta\int L^+_{\rm Dir}(\zeta)=0$ and formula (46) implies that $\delta\int L(\zeta)=0$. As the latter is true for an arbitrary variation of ζ this means that ζ is a solution of the field equation for the Lagrangian density L.

Suppose that ζ is a solution of the field equation for the Lagrangian density L. Then $L(\zeta)=0$ and formula (41) implies that either $L^+_{\rm Dir}(\zeta)=0$ or $L^-_{\rm Dir}(\zeta)=0$; note that in view of (42) we cannot have simultaneously $L^+_{\rm Dir}(\zeta)=0$ and $L^-_{\rm Dir}(\zeta)=0$. Assume for definiteness that $L^+_{\rm Dir}(\zeta)=0$. [The case when $L^-_{\rm Dir}(\zeta)=0$ is handled similarly.] Varying ζ and repeating the argument from the previous paragraph we arrive at (46). We assumed that ζ is a solution of the field equation for the Lagrangian density L so $\delta\int L(\zeta)=0$ and formula (46) implies that $\delta\int L^+_{\rm Dir}(\zeta)=0$. As the latter is true for an arbitrary variation

of ζ this means that ζ is a solution of the field equation for the Lagrangian density L^+_{Dir}. $\square$

The proof of Theorem 1 presented above may appear to be non-rigorous but it can be easily recast in terms of explicitly written field equations.

Theorem 1 establishes that in the quasi-stationary case our model reduces to a pair of massless Dirac equations (6). There is, however, a small logical flaw in this statement. The full time-dependent field equations for our model (as well as for the massless Dirac model) are obtained by varying the spinor field ξ by a $\delta\xi$ with arbitrary dependence on time whereas in the proof of Theorem 1 we have effectively varied the spinor field ξ maintaining quasi-stationarity, i.e. we took

$$\delta\xi^a(x^0,x^1,x^2,x^3)=e^{-i\omega x^0}\delta\zeta^a(x^1,x^2,x^3) \tag{47}$$

(compare with (30)). Note that the $\delta\zeta$ in the above formula does not depend on time. If we now modify (47) so that $\delta\zeta$ depends on time this will generate an extra term in the field equations, one with the time derivative. It turns out that this extra term with the time derivative vanishes. For the sake of brevity we do not present the corresponding calculation.

8 Plane wave solutions

Suppose that $M=\mathbb{R}^3$ is Euclidean 3-space equipped with Cartesian coordinates $x=(x^1,x^2,x^3)$ and metric $g_{\alpha\beta}=\mathrm{diag}(-1,-1,-1)$. Let us choose constant Pauli matrices

$$\sigma_{\alpha a\dot b}=\begin{pmatrix}\sigma_{1a\dot b}\\ \sigma_{2a\dot b}\\ \sigma_{3a\dot b}\end{pmatrix}:=\begin{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ \begin{pmatrix}0&i\\-i&0\end{pmatrix}\\ \begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{pmatrix} \tag{48}$$

(compare with (36), (37)) and seek solutions of the form

$$\xi^a(x^0,x^1,x^2,x^3)=e^{-i(\omega x^0+k\cdot x)}\zeta^a \tag{49}$$

(compare with (30)) where $\omega\neq 0$ is a real number, $k=(k_1,k_2,k_3)$ is a real constant covector and $\zeta\neq 0$ is a (complex) constant spinor. We shall call solutions of the type (49) *plane wave*. In seeking plane wave solutions what we are doing is separating out all the variables, namely, the time variable x^0 and the spatial variables $x=(x^1,x^2,x^3)$.

We look at the field equations of our model described in Section 4. These field equations are highly nonlinear so it is not *a priori* clear that one can seek solutions in the form of plane waves. However, plane wave solutions are a special case of quasi-stationary solutions and the latter were analysed in Section 7. We

know (Theorem 1) that in the quasi-stationary case our model reduces to a pair of Dirac equations (6). Substituting (2), (48) and (49) into (6) we get

$$\begin{pmatrix} \pm\omega - k_3 & -k_1 - ik_2 \\ -k_1 + ik_2 & \pm\omega + k_3 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{50}$$

The determinant of the matrix in the LHS of (50) is $\omega^2 - k_1^2 - k_2^2 - k_3^2$ so this system has a nontrivial solution ζ if and only if $k_1^2 + k_2^2 + k_3^2 = \omega^2$. Our model is invariant under rotations of the Cartesian coordinate system (orthogonal transformations of the coordinate system which preserve orientation), so, without loss of generality, we can assume that

$$k_\alpha = \begin{pmatrix} 0 \\ 0 \\ \pm\omega \end{pmatrix}. \tag{51}$$

Substituting (51) into (50) we conclude that, up to scaling by a nonzero complex factor, we have

$$\zeta^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{52}$$

Combining formulae (49), (51) and (52) we conclude that for each real $\omega \neq 0$ our model admits, up to a rotation of the coordinate system and rescaling, two plane wave solutions and that these plane wave solutions are given by the explicit formula

$$\xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega(x^0 \pm x^3)}. \tag{53}$$

Let us now rewrite the plane wave solutions (53) in terms of our original dynamical variables, coframe ϑ and density ρ. Substituting (2), (48) and (53) into (25)–(29) we get $\rho = 1$ and

$$\vartheta^1_\alpha = \begin{pmatrix} \cos 2\omega(x^0 \pm x^3) \\ -\sin 2\omega(x^0 \pm x^3) \\ 0 \end{pmatrix}, \quad \vartheta^2_\alpha = \begin{pmatrix} \sin 2\omega(x^0 \pm x^3) \\ \cos 2\omega(x^0 \pm x^3) \\ 0 \end{pmatrix}, \quad \vartheta^3_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{54}$$

Note that scaling of the spinor ζ by a nonzero complex factor is equivalent to scaling of the density ρ by a positive real factor and time shift $x^0 \mapsto x^0 + \text{const}$.

We will now establish how many different (ones that cannot be continuously transformed into one another) plane wave solutions we have. To this end, we rewrite formula (54) in the form

$$\vartheta^1_\alpha = \begin{pmatrix} \cos 2|\omega|(x^0 + bx^3) \\ -a \sin 2|\omega|(x^0 + bx^3) \\ 0 \end{pmatrix}, \ \vartheta^2_\alpha = \begin{pmatrix} a \sin 2|\omega|(x^0 + bx^3) \\ \cos 2|\omega|(x^0 + bx^3) \\ 0 \end{pmatrix}, \ \vartheta^3_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{55}$$

where a and b can, independently, take values ± 1. It may seem that we have a total of 4 different plane wave solutions. Recall, however (see Remark 1), that we can perform rigid rotations of the coframe and that we have agreed to view

coframes that differ by a rigid rotation as equivalent. Let us perform a rotation of the coordinate system

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ x^1 \\ -x^3 \end{pmatrix}$$

simultaneously with a rotation of the coframe

$$\begin{pmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \end{pmatrix} \mapsto \begin{pmatrix} \vartheta^2 \\ \vartheta^1 \\ -\vartheta^3 \end{pmatrix}.$$

It is easy to see that the above transformations turn a solution of the form (55) into a solution of this form again only with

$$a \mapsto -a, \qquad b \mapsto -b.$$

Thus, the numbers a and b on their own do not characterise different plane wave solutions. Different plane wave solutions are characterised by the number $c := ab$ which can take two values, $+1$ and -1.

The bottom line is that we have two essentially different types of plane wave solutions. These can be written, for example, as

$$\vartheta^1_{\ \alpha} = \begin{pmatrix} \cos 2|\omega|(x^0 + x^3) \\ \mp \sin 2|\omega|(x^0 + x^3) \\ 0 \end{pmatrix}, \quad \vartheta^2_{\ \alpha} = \begin{pmatrix} \pm \sin 2|\omega|(x^0 + x^3) \\ \cos 2|\omega|(x^0 + x^3) \\ 0 \end{pmatrix}, \quad \vartheta^3_{\ \alpha} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{56}$$

The plane wave solutions (56) describe travelling waves of rotations. Both waves travel with the same velocity in the negative x^3-direction. The difference between the two solutions is in the direction of rotation of the coframe: if we fix x^3 and look at the evolution of (56) as a function of time x^0 then one solution describes a clockwise rotation whereas the other solution describes an anticlockwise rotation. We identify one of the solutions (56) with a left-handed neutrino and the other with a right-handed antineutrino.

9 Discussion

As explained in Section 4, our mathematical model is a special case of the theory of teleparallelism which in turn is a special case of Cosserat elasticity.

The differences between our mathematical model formulated in Section 4 and mathematical models commonly used in teleparallelism are as follows.

- We assume the metric to be prescribed (fixed) whereas in teleparallelism it is traditional to view the metric as a dynamical variable. In other words, in works on teleparallelism it is customary to view (8) not as a constraint but as a definition of the metric and, consequently, to vary the coframe without any constraints at all. This is not surprising as most, if not all, authors who contributed to teleparallelism came to the subject from General Relativity.

- We choose a very particular Lagrangian density (18) containing only one irreducible piece of torsion (axial) whereas in teleparallelism it is traditional to choose a more general Lagrangian containing all three pieces (tensor, trace and axial), see formula (26) in [12]. In choosing our particular Lagrangian density (18) we were guided by the principle of conformal invariance.

The main result of our paper is Theorem 1 which establishes that in the quasi-stationary case (prescribed oscillation in time with frequency ω) our mathematical model is equivalent to a pair of massless Dirac equations (6).

This leaves us with two issues.

A What can be said about the general case, when the the spinor field ξ is an arbitrary function of all spacetime coordinates (x^0, x^1, x^2, x^3) and is not necessarily of the form (30)?

B What can be said about the relativistic version of our model described in Section 5?

The two issues are, of course, related: both arise because in formulating our basic model in Section 4 we adopted the Newtonian approach which specifies the time coordinate x^0 ("absolute time").

We plan to tackle issue A by means of perturbation theory. Namely, assuming the metric to be flat (as in Section 8), we start with a plane wave (49) and then seek the unknown spinor field ξ in the form

$$\xi^a(x^0, x^1, x^2, x^3) = e^{-i(\omega x^0 + k\cdot x)}\zeta^a(x^0, x^1, x^2, x^3) \tag{57}$$

where ζ is a slowly varying spinor field. Here "slowly varying" means that second derivatives of ζ can be neglected compared to the first. Our conjecture is that the application of a formal perturbation argument will yield a massless Dirac equation for the spinor field ξ.

We plan to tackle issue B by means of perturbation theory as well. The relativistic version of our model has 3 extra field equations corresponding to the 3 extra dynamical degrees of freedom (Lorentz boosts in 3 directions). Our conjecture is that if we take a solution of the nonrelativistic problem which is a perturbation of a plane wave (as in the previous paragraph) then, at a perturbative level, this solution will automatically satisfy the 3 extra field equations. In other words, we conjecture that our nonrelativistic model possesses relativistic invariance at the perturbative level.

The detailed analysis of the two issues flagged up above will be the subject of a separate paper.

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